Modified Gram-Schmidt Process

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- Replace the text so that it reads as follows from the bottom of page 437, line −10, through the top of p. 438 ending at **EXAMPLE 3**:

Schmidt process can be stabilized. Instead of computing the vector $\mathbf{z}_k$ as above, we compute them as

$$\begin{aligned}
\mathbf{z}_k^{(1)} &\leftarrow \mathbf{v}_k - \text{proj}_{\mathbf{z}_1}\mathbf{v}_k \\
\mathbf{z}_k^{(2)} &\leftarrow \mathbf{z}_k^{(1)} - \text{proj}_{\mathbf{z}_2}\mathbf{z}_k^{(1)} \\
\mathbf{z}_k^{(3)} &\leftarrow \mathbf{z}_k^{(2)} - \text{proj}_{\mathbf{z}_3}\mathbf{z}_k^{(2)} \\
&\vdots \\
\mathbf{z}_k^{(k-2)} &\leftarrow \mathbf{z}_k^{(k-3)} - \text{proj}_{\mathbf{z}_{k-2}}\mathbf{z}_k^{(k-3)} \\
\mathbf{z}_k^{(k-1)} &\leftarrow \mathbf{z}_k^{(k-2)} - \text{proj}_{\mathbf{z}_{k-1}}\mathbf{z}_k^{(k-2)}
\end{aligned}$$

Each step finds a vector $\mathbf{z}_k^{(i)}$ orthogonal to $\mathbf{z}_k^{(i-1)}$. Thus, $\mathbf{z}_k^{(i)}$ is also orthogonalized against any error introduced in the computation of $\mathbf{z}_k^{(i-1)}$. In exact arithmetic, this computation gives the same result as the original form, but it produces smaller errors in finite-precision computer arithmetic. A computer algorithm for the modified Gram-Schmidt process follows:

for $i = 1$ **to** k
  $\mathbf{v}_i \leftarrow (1/||\mathbf{v}_i||)\mathbf{v}_i$ (normalized)
  for $j = 1$ **to** $j - 1$
    $\mathbf{v}_j \leftarrow \mathbf{v}_j - \text{proj}_{\mathbf{v}_i}\mathbf{v}_j$ (remove component in direction $\mathbf{v}_i$)
  end for
end for

Here the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ are replaces by an orthonormal set of vectors that space the same subspace. The cost of this algorithm is asymptotically $2nk^2$ floating-point operations where n is the dimensionality of the vectors.